

## Properties of r.e. Languages and Rice Theorem

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A *property* of the r.e. languages is a set of r.e. languages. Thus, the property of being regular is the set of all regular languages, and the property of being empty is the set  $\{\emptyset\}$ .

A property  $\mathcal{P}$  is *trivial* if it is either empty (i.e. satisfied by no language at all), or is all r.e. languages.

Since a typical language  $L$  could be infinite,  $L$  itself cannot be written down as an input to a Turing machine. Thus we need to find “a finite representation” of  $L$ . A natural choice would be to consider a TM  $M$  that accepts  $L$ . The description (encoding) of  $M$  is finite, although  $L(M)$  is infinite. Since there are many TM's that accept  $L$ , we will consider all of them. Thus, if  $\mathcal{P}$  is a property of the r.e. languages, the languages  $L_{\mathcal{P}}$  is the set of codes for Turing machines  $\langle M \rangle$  such that  $L(\langle M \rangle) \in \mathcal{P}$ . When we talk about the decidability of a property  $\mathcal{P}$ , we mean the decidability of  $L_{\mathcal{P}}$ .

**Theorem 1 (Rice's Theorem)** *Every nontrivial property of the r.e. languages is undecidable.*

**Proof.** Let  $\mathcal{P}$  be a nontrivial property of the r.e. languages. First we assume that  $\emptyset \notin \mathcal{P}$ . Since  $\mathcal{P} \neq \emptyset$ , there must be an r.e. language  $L \in \mathcal{P}$  such that  $L \neq \emptyset$ . Let  $M_L$  be a TM accepting  $L$ .

We shall reduce the halting problem  $H$  to  $L_{\mathcal{P}}$ . Recall that

$$H = \{\langle M, x \rangle \mid M \text{ is a TM and halts on input } x\},$$

and that  $H$  is undecidable.

We now construct a reduction  $f$ . On input  $\langle M, x \rangle$ , the reduction  $f$  outputs a TM  $\langle M' \rangle$  such that on any input  $w$ ,  $M'$  first simulates  $M$  on  $x$ . If  $M$  halts on input  $x$ , then  $M'$  simulates  $M_L$  on  $w$ . That is,  $f(\langle M, x \rangle) = \langle M' \rangle$ .

We verify that  $f$  is indeed a reduction. It is easy to see that  $f$  is computable. For any  $\langle M, x \rangle$ , if  $\langle M, x \rangle \in H$ , then  $L(M') = L(M_L) = L \in \mathcal{P}$ , and so  $\langle M' \rangle \in L_{\mathcal{P}}$ . Conversely, if  $\langle M, x \rangle \notin H$ , then  $L(M') = \emptyset \notin \mathcal{P}$ , and so  $\langle M' \rangle \notin L_{\mathcal{P}}$ . Thus, we have

$$\langle M, x \rangle \in H \text{ iff } f(\langle M, x \rangle) \in L_{\mathcal{P}}.$$

This establishes that  $L_{\mathcal{P}}$  is not decidable. Now we assume that  $\emptyset \in \mathcal{P}$ , then  $\emptyset \notin \overline{\mathcal{P}}$ . Hence, using the above proof we know that  $L_{\overline{\mathcal{P}}}$  is undecidable. Since

$$L_{\overline{\mathcal{P}}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \in \overline{\mathcal{P}}\},$$

and

$$\overline{L_{\overline{\mathcal{P}}}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \notin \overline{\mathcal{P}}\},$$

we have  $\overline{L_{\overline{\mathcal{P}}}} = L_{\mathcal{P}}$  is undecidable, and so  $L_{\mathcal{P}}$  is undecidable. ■

It is important to note that Rice Theorem only applies to sets on properties of general Turing machines. If restrictions are added, then Rice Theorem may no longer apply. For instance, Rice Theorem does not apply to the following set:

$$R_{NE} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) \neq \emptyset\}.$$

This set is on properties of DFA's. In fact,  $R_{NE}$  is decidable: For any given DFA  $M$ , we can check whether there is a final state that is reachable from the start state. If so, then  $L(M) \neq \emptyset$ ; otherwise,  $L(M) = \emptyset$ . But Rice Theorem applies to the following set, and so it is undecidable.

$$T_{NE} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \neq \emptyset\}.$$