1. Consider the following array of integers:

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<thead>
<tr>
<th>Element# / Pass j =?</th>
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Show how insertion sort would sort this array.

2. For any real constants $a$ and $b$, where $b > 0$, show that $(n + a)^b = \Theta(n^b)$. 
\[(n + a)^b = \Theta(n^b)\]

We must show that:
\[c_1 n^b \leq (n + a)^b \leq c_2 n^b,\]
for any constants \(a, b, c_1, c_2,\) where the last three are positive. Consider \(n + a.\)

\[n + a \leq n + |a| \leq 2n, \text{ when } |a| \leq n.\]

Also we have
\[n - a \geq n - |a| \geq \frac{1}{2} n, \text{ when } \frac{1}{2} n \geq |a|\]

So, when \(n \geq 2|a|:\)
\[
\frac{1}{2} n \leq n + a \leq 2n
\]

and since \(b > 0\)
\[
\left(\frac{1}{2}\right)^b = \left(\frac{1}{2}\right)^b \leq (n + a)^b \leq (2n)^b
\]
\[
\frac{1}{2^b} n^b \leq (n + a)^b \leq 2^b n^b
\]

Let \(c_1 = \frac{1}{2^b}\) and \(c_2 = 2^b\)

3. Show that \(o\left(g(n)\right) \cap \omega\left(g(n)\right) = \emptyset.\)

Suppose \(o\left(g(n)\right) \cap \omega\left(g(n)\right) \neq \emptyset.\) Then there exists some \(f(n)\) such that \(c_1 g(n) < f(n) < c_2 g(n),\) for all \(c_1\) and \(c_2.\) But if this inequality holds for all such constants, then \(c g(n) < f(n) < c g(n),\) for any constant \(c\) and all \(n > n_0,\) for some \(n_0.\) But since \(c > 0,\) this is impossible.

4. Which of the following statements are true? Justify your answer:

a) \(n^3 \in O(n^5)\) True, you can find a \(c > 0\) such that \(n^3 < cn^5\)

b) \(n^{2.5} \in \Omega(n^2)\) True, there is a \(c > 0\) such that \(n^{2.5} > cn^2\)
c) $4^n \in \Theta(2^n)$. True. $4^n = (2^2)^n = 2^{2n} = 2^2 \cdot 2^n = 4 \cdot 2^n$. Since 4 is a constant, this works to establish the tight bound.

d) $\left(\frac{3}{2}\right)^n \in O(2^n)$. True, $c>1$ works for all $n$.

e) Is $n^{\lg c} = O(c^{\lg n})$? By log identity, $n^{\lg c} = c^{\lg n}$ and so it’s true for any $c$ and $n$.

5. Is $f(n) = O\left(\left(\frac{f(n)}{n}\right)^2\right)$? Why or why not? Not true. Consider $f(n) = 1/n$. $1/n > (1/n)^2$ for $n > 2$.

6. Consider the recurrence: $T(n) = T(n - 1) + 1, T(0) = 0$. Use the substitution technique shown in class to solve the recurrence.

\[
T(n) = T(n - 1) + 1 \\
= T(n - 2) + 1 + 1 \\
= T(n - 3) + 3 \\
= T(n - 4) + 4 \\
\vdots \\
= T(n - i) + i \\
= T(0) + n = n
\]