Symbolic Computation Techniques and Stability Properties of Network Models with Delay

by
G. Pecelli and B. G. Kim
Department of Computer Science
University of Massachusetts - Lowell
Lowell, MA 01854
kim@cs.uml.edu, giam@cs.uml.edu

1. Introduction. We introduce some simple network models that attempt to satisfy two, possibly contradictory, requirements. On the one hand, they attempt to be realizable models of network traffic addressing issues of starvation, buffer size control, transmission delays and user populations distributed over several delay zones, while, on the other hand, they attempt to be mathematically tractable. Mathematical tractability is interpreted as smoothness plus the existence of appropriate constant solutions whose stability properties can be fully analyzed, and whose transition to instability can be also analyzed via techniques associated with the Hopf Bifurcation Theorem. Symbolic computation techniques using Maple are applied to the computation of complex constants of the system required for the Hopf results.

2. The General Models.
Let \( \{T_i: i = 1, \ldots, n\} \) and \( \{r_i: i = 1, \ldots, n\} \) be sequences of non-negative numbers; let \( \{a_i: i = 1, \ldots, n\} \) and \( \{k_i: i = 1, \ldots, n\} \) be sequences of positive numbers; let \( c \) and \( y_0 \) be positive numbers.

Consider the system of \( n + 1 \) delay-differential equations

\[
\begin{align*}
\frac{dX_i(t)}{dt} &= a_i \left(1 - \frac{X_i(t)}{k_i}\right) - X_i(t) F_i(Y(t), Y'(t)) \quad \text{for } i = 1\ldots n \\
\frac{dY(t)}{dt} &= \begin{cases} 
\sum_{j=1}^{n} r_j X_j(t - T) - c & \text{if } Y(t) > 0 \text{ or the sum is } > 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

We interpret \( i \) as denoting a "user zone", with \( T_i \) the delay from zone \( i \) to the buffer (the delay can be split into direct and return, but a simple time shift along the solutions will give (2.1)); \( X_i \) is the transmission rate from zone \( i \); \( a_i \) is the maximum rate of increase of the transmission rate; \( k_i \) is the
maximum sustainable transmission rate; \( r_i \) is the “number of users” in zone \( i \); c is the rate at which the buffer can be emptied; \( F_i \) provides the squelch rate as a function of buffer use and the rate of change of buffer use.

A number of general results can be obtained on the existence and stability of positive constant solutions. Unfortunately, the characteristic equations are polynomials in both \( \lambda \) and \( e^\lambda \), and the analysis becomes quite difficult even for simple systems.

The simplest model, and the only one to have been “completely” analyzed is the one with a single zone - we can think of it as representing some kind of “average” of a multi-zone system.

The class of models (2.1) is being proposed in lieu of the classical model ([Yin 94] and many others):

\[
\begin{bmatrix}
X'(t) \\
Y'(t)
\end{bmatrix} =
\begin{cases}
\begin{align*}
a & \text{if } Y(t) \leq y_0 \\
-b & \text{otherwise}
\end{align*}
\end{cases}
\begin{cases}
\begin{align*}
X(t - 1) + r X(t - m) - c & \text{if } Y(t) > 0 \text{ or the sum is } > 0 \\
0 & \text{otherwise}
\end{align*}
\end{cases}
\]

which has been shown in [Pecelli 95] to have a number of undesirable asymptotic properties.


We start by examining the simplest configuration: \( r_1 = r \), \( r_j = 0 \), for \( j = 2, \ldots, n \), so that we have a single zone. We assume, furthermore, that \( F \) depends only on \( Y \) and not \( Y' \). This is not very interesting, but it is the easiest case, and the one where a complete analysis is possible. Linearizing around the fixed point \((X^*, Y^*) = (\frac{c}{r}, F^{-1}(\frac{a}{c \lambda} (r \lambda - c))) \), which, if \( \lambda r > c \) and \( F \) is reasonable, will have both positive components, and re-normalizing by \( t = T \tau \), we obtain the equivalent second order DDE \((F_1 = F_Y(Y^*))\):

\[
Y''(\tau) + \frac{a r T}{c} Y'(\tau) + T^2 c F_1 Y(\tau - 1)) = 0.
\]

The characteristic equation for this configuration is

\[
\lambda^2 + \frac{a r T}{c} \lambda + T^2 c F_1 \exp(- \lambda) = 0.
\]
If the original equation had been unretarded \((T = 0)\), we would have the algebraic characteristic equation 
\[
\lambda^2 + \frac{a r}{c} \lambda + c F_1 = 0,
\]
with the solutions
\[
\lambda_{1,2} = \frac{-ar \pm ((ar)^2 - 4 c^3 F_1^{1/2})}{2 c}
\]
implying an asymptotically stable constant solution regardless of the values of the (positive) parameters.

Let \(a(a, c, r, T) = a r T c\), \(b(c, F_1, T) = T^2 c F_1\), so that the characteristic equation becomes
\[(3.1) \quad \lambda^2 + a(a, c, r, T) \lambda + b(c, F_1, T) \exp(-\lambda) = 0.
\]

To deal with (3.1) we need the following:

**Lemma 3.1** Assume that \(a(r)\) and \(b(r)\) are positive and smooth (with smooth inverses if and where needed) over their common domains of definition.

a) All roots \(\lambda(r) = \xi(r) + i\eta(r)\) of (*) have negative real parts if and only if \(a(r)/b(r) > (\sin \xi(r))/\xi(r)\) where \(\xi(r)\) is the unique root of \(\xi = a(r)\cot \xi\) or, equivalently, of \(\xi^2 = b(r) \cos \xi\), \(0 < \xi < \pi/2\).

b) In the first quadrant of \((a, b)\)-space the boundary of the region of stability
   \(B(\xi) = (\xi \tan \xi, \xi^2 \sec \xi), 0 \leq \xi < \pi/2,\) has the properties
   i) \(\frac{d\beta}{d\alpha} > 0, \lim \alpha \to 0^+ \frac{d\beta}{d\alpha} = 1,\) and \(\lim \alpha \to \infty \frac{d\beta}{d\alpha} = \pi/2\) on \(B(\xi)\),
   ii) \(\frac{d^2\beta}{d\alpha^2} > 0, \lim \alpha \to 0^+ \frac{d^2\beta}{d\alpha^2} = 1/3,\) and \(\lim \alpha \to \infty \frac{d^2\beta}{d\alpha^2} = 0\) on \(B(\xi)\).

c) Assume that \((a(r), b(r))\) is a curve that crosses \(B(\xi)\) from the stable to the unstable region at \(r = r_0\) (and thus \(\xi = \eta(r_0)\)). Then
   \[
   \xi'(r_0) = -\cos(\xi) \left(\frac{\xi (\sin(\xi) + 2 \cos(\xi))}{\xi (\xi^2 + 2 \xi \sin(\xi) \cos(\xi) + 3 \cos(\xi)^2 + 1)}\right) \frac{\alpha'(r_0)}{\beta'(r_0)} - (\sin(\xi) \cos(\xi) + \xi) \beta'(r_0)
   \]

d) If \(\beta(r) < \max(4\pi^2, 2\pi a(r))\) and \(\xi'(r_0) > 0\) if \((\alpha(r_0), \beta(r_0))\) is on \(B(\xi)\). (3.1) has at most two roots with positive real parts, counted with their multiplicities.

**Proof.** [Pecelli 94] - just grind away at the expressions. Maple was used (and needed) to help the book-keeping.
This gives the LINEAR STABILITY RESULT:

**THEOREM 3.2.** If \( \frac{\beta(c, F_1, T)}{\alpha(a, c, r, T)} = \frac{c^2 T F_1}{a r} < 1 \), then the constant solution \((X^*, Y^*)\) is asymptotically stable. If \( \frac{c^2 T F_1}{a r} > \frac{\pi}{2} \), then the constant solution is unstable. For values of \( \frac{c^2 T F_1}{a r} \) in \([1, \pi/2]\), the system could be either stable or unstable - a more precise positioning of the parameter "curve" is needed, but the stability boundary is concave up and lies entirely between the lines \( \beta = \alpha \) and \( \beta = \frac{\pi}{2} \alpha \). Furthermore, as \( T \) increases, the constant solution exhibits a Hopf Bifurcation as its stability status changes from stable to unstable.

To apply the Hopf Bifurcation Theorem [Hale 77] - and thus conclude that a family of small amplitude periodic orbits "surrounds" the constant solution either before (w.r.t. \( T \) - subcritical bifurcation), during, or after (supercritical bifurcation) the constant solution has become unstable - we need to show that some form of condition c) of the Lemma is satisfied - the parameter curve must cross the stability boundary curve transversally. Using \( T \) as the bifurcation parameter, if \( \lambda(T_0) = \xi(T_0) + i\eta(T_0) \) denotes the rightmost zero with positive imaginary part of the characteristic equation as a function of \( T \), we need to show that \( \xi'(T_0) > 0 \) for \( \xi(T_0) = 0 \).

The geometry - \((\alpha(T), \beta(T))\) is a parabola parametrized by \( T \), while the stability boundary lies between the two lines \( \beta = \alpha \) and \( \beta = \frac{\pi}{2} \alpha \) - already tells us that we can expect a transversal intersection. To verify it, we compute (with Maple's assistance, and at \( T = T_0, \eta = \eta(T_0) \)), using (3.2) below,

\[
\xi'(T) = \frac{\eta^2 (1 + \cos(\eta)^2)}{T (2 \eta \sin(\eta) \cos(\eta) + 3 \cos(\eta)^2 + \eta^2 + 1)},
\]

which is clearly positive for all \( \eta \) in \((0, \pi/2)\).

We thus have the existence of a family of periodic orbits bifurcating from the constant solution. Unfortunately, the linear analysis is insufficient to
determine the stability properties of these orbits. A DESIRABLE conclusion would be that a family of asymptotically stable orbits appears in small neighborhoods of the constant solution immediately after transition to instability. An UNDESIRABLE conclusion would be that the family appears before transition and is unstable, since this would indicate that the network behavior might change abruptly at transition. It may, in this case, exhibit undesirable characteristics well before transition: orbits near, but not very near the constant one may well be captured by “large amplitude” periodic orbits existing “outside” the unstable ones: the “basin of attraction” for the constant solution may be very small. We will use the techniques presented in [Chow 77] and realized in software in [Pecelli 91, 93] for the non-linear analysis.

**THEOREM 3.3.** With \( \eta \) = positive imaginary part of the crossing eigenvalue, and with \( \nu \) the parameter for which transition to instability occurs at \( \nu = 0 \), we can scale and transform the proper variant of (2.1) into the “averaged” system

\[
\begin{align*}
\dot{r} &= \epsilon v r + \epsilon^2 r^3 K + O(\epsilon^2 \nu) + O(\epsilon^3) \\
\dot{\zeta} &= -\eta(\nu) + O(\epsilon) \\
\dot{y}_t &= A Q(\nu) y_t + O(\epsilon)
\end{align*}
\]

where the first two equations are ODEs over \( \mathbb{R}^2 \), and where \( K = K^* + K^{**} \), with

\[
K^* = \frac{1}{2\pi} \int_0^{2\pi} [C_4(\zeta, 0) + \frac{1}{\eta} C_3(\zeta, 0) D_3(\zeta, 0)] d\zeta \quad \text{and}
\]

\[
K^{**} = \frac{1}{2\pi} \int_0^{2\pi} w^*(\zeta) J(0) \left[ \frac{\cos \zeta}{\sin \zeta} \right]^2 d\zeta,
\]

where \( C_3(\zeta, 0), C_4(\zeta, 0) \) and \( D_3(\zeta, 0) \) are trigonometric polynomials in \( \zeta \) of degrees 3, 4 and 3 arising from the transformation; \( w^*(\zeta) \) is the unique 2\pi-periodic solution of \( G_2(\zeta, 0) - \eta w^*(\zeta) + w^*(\zeta) A Q = 0 \), with \( G_2(\zeta, 0), w^*(\zeta), w^*(\zeta) \) and \( A Q \in L(Q_{\lambda(0)}, \mathbb{R}) \) appropriate linear operators on certain infinite dimensional Banach spaces, and \( Q_{\lambda(0)} \) is the ‘orthogonal subspace’ (w.r.t. a certain bilinear form) for the 2-dimensional eigensolution space corresponding to \( \nu = 0 \).

**Proof.** [Chow 77]. \( \nu = T - T_0 \).
If we now scale $v$ once more and recall that $y_t$ will remain small for all future time, the choice $v = - \text{sgn}(K) \varepsilon$ leads to
\[ r' = \varepsilon^2(\pm r + r^3 K) + O(\varepsilon^3), \text{ where } \pm = - \text{sgn}(K). \]

This indicates we should have a periodic solution roughly at $r = |K|^{-1/2}$, which would mean, in the original (unscaled) coordinates, with $v = - \text{sgn}(K) \varepsilon^2$
\[
\begin{align*}
& r = \varepsilon |K|^{-1/2} + O(\varepsilon^2) \\
& \zeta = - \eta t + O(\varepsilon) \\
& y_t = O(\varepsilon^2). 
\end{align*}
\]

We can thus read both amplitude and period of the bifurcating solution right off the averaged system.

One can show (Chow and Mallet-Paret, again) that $K < 0$ leads to a stable periodic solution existing for $v > 0$, while $K > 0$ leads to an unstable periodic solution existing for $v < 0$. The case $K = 0$ is the non-generic case, requiring further computation for the determination of the character of the bifurcation.

[\textit{Pecelli 91, 93}] details the software (in Maple) constructed to perform the computations indicated by Theorem 3.3.

Using this theorem and the software we can compute $K$ and conclude:

**Theorem 3.4.** Assume that $F''(Y^*) \leq 0$ and $F'''(Y^*) \leq 0$. Let $a$, $c$, $\kappa$ and $r$ be given, all positive and such that $\kappa r > c$. Then there exists a unique value $T_0$ of $T$ such that the constant solution $(X^*, Y^*)$ (dependent on $T_0$) is asymptotically stable for $T < T_0$ and unstable for $T > T_0$. Furthermore, as $T$ increases beyond $T_0$, the system exhibits a supercritical Hopf Bifurcation, i.e. a one-parameter family of asymptotically stable periodic orbits whose amplitude is approximately proportional to $|T - T_0|$, and whose period is approximately \[ \frac{2\pi}{\text{Im}(\lambda(T_0))}, \text{ where } \lambda(T_0) \text{ is the eigenvalue of positive imaginary part that lies on the imaginary axis. Generically, these orbits are the only periodic orbits of small amplitude in a neighborhood of the constant solution.} \]

**Proof.** The characteristic equation leads to the requirement that
\[(3.2) \begin{align*}
&c F_1 T^2 \cos(\eta) - \eta^2 = 0 \\
&c^2 F_1 T \sin(\eta) - a r \eta = 0
\end{align*}\]

which we can use to compute \(a\) and \(F_1\) in terms of the remaining parameters. We choose to keep \(T\) and \(r\) explicit, since \(T\) is the parameter whose change leads to the bifurcation, while \(r\) indicates the amount of traffic the system supports. We make use of the software described in [Pecelli 93] to compute the bifurcation constant

\[
K = K(r, c, T, \eta, F_2, F_3)
\]

\[
= r^2 \frac{n_0(\eta) + n_3(\eta) c^2 F_2 T^3 + n_4(\eta) c^3 F_3 T^4 + n_6(\eta) c^4 F_2^2 T^6}{d(\eta)},
\]

where \(F_2 = F''(Y^*), F_3 = F'''(Y^*)\),

\[
d(\eta) = 8 c^2 \eta^5 (8 \cos(\eta)^3 - 12 \cos(\eta)^2 - 5)
\]

\[
(1 + \eta^2 + 2 \eta \cos(\eta) \sin(\eta) + 3 \cos(\eta)^2) < 0 \text{ on } (0, \pi/2).
\]

We plot the numerator expressions to conclude:

\[
n_0(\eta) = 2 \eta^6 ((16 \cos(\eta)^5 - 24 \cos(\eta)^4 - 28 \cos(\eta)^3 + 26 \cos(\eta)^2 + 6 \cos(\eta) + 7) \eta
\]

\[
- \sin(\eta) (40 \cos(\eta)^4 - 48 \cos(\eta)^3 - 6 \cos(\eta)^2 - 5 \cos(\eta)
\]

\[
+ 4)) > 0, \text{ on } (0, \pi/2).
\]

This implies that a linear squelching function will always give rise to a supercritical bifurcation. The system will exhibit stable periodic orbits of small amplitude bifurcating from the unstable constant solution.

\[
n_3(\eta) = 2 \eta^3 \cos(\eta)
\]

\[
(\sin(\eta) (16 \cos(\eta)^3 - 18 \cos(\eta)^2 - 3 \cos(\eta) - 7) \eta
\]

\[
+ 2 (1 - \cos(\eta)) (8 \cos(\eta)^5 + 2 \cos(\eta)^4 - 8 \cos(\eta)^3 - \cos(\eta) + 2))
\]

\[
< 0, \text{ on } (0, \pi/2).
\]

\[
n_4(\eta) = \eta^2 \cos(\eta) (8 \cos(\eta)^3 - 12 \cos(\eta)^2 - 5) (\cos(\eta) \sin(\eta) + \eta) < 0
\]

\[
on (0, \pi/2).
\]

\[
n_6(\eta) = \cos(\eta)^2 ((11 + 24 \cos(\eta)^2 - 20 \cos(\eta)^3) \eta
\]

\[
- \sin(\eta) (24 \cos(\eta)^4 - 24 \cos(\eta)^3 + 6 \cos(\eta)^2
\]

\[
- 11 \cos(\eta) + 2)) > 0, \text{ on } (0, \pi/2).
\]

And the theorem follows.

If \(F''(Y^*) \leq 0\) and \(F'''(Y^*) \leq 0\), the bifurcation is supercritical (stable periodic orbits appearing after the constant solution has become unstable) for all values of \(T\) (the total feedback delay at bifurcation), while it may be subcritical otherwise (unstable periodic orbits appearing before the constant...
solution has become unstable). We thus conclude that linear squelching functions are always "safe", while non-linear ones may give rise to undesirable behavior: transitioning into instability without a small amplitude stable periodic orbit to keep the oscillations controlled might mean an abrupt transition to a "large amplitude" orbit or to more complex behavior, with concomitant lack of predictability for buffer requirements and minimum transmission rates.

4. **Some final observations:**
   a) For any given choices for \( a, c, r \) and \( F_1 \), if the total feedback delay satisfies \( T < \frac{a r}{c^2 F_1} \), then the constant solution is asymptotically stable.

   This tells us that, for this family of delay equations, the introduction of a small delay has a small effect on the stability properties. On the other hand, it also tells us that, as soon as a feedback delay is introduced, there are parameter configurations for which the constant solution is unstable. [Pecelli 94] presents examples with a biological motivation.

   b) Given a stable configuration, INCREASING the values of \( a \) and \( r \), and thus of the rate of transmission increase and the number of transmitting sites will NOT destabilize the system. This appears to be counterintuitive, since it appears to say that increasing the rate at which traffic enters the system will, in fact, lead to stabler systems...

   c) Given a stable configuration, increasing the feedback delay will lead to a system with an unstable constant solution. This is not entirely surprising.

   d) Attempting to introduce a more "forceful" damping (i.e., making \( F_1 \) larger) will eventually destabilize an asymptotically stable constant solution. Furthermore, increasing the buffer flush rate \( c \) may destabilize the system, unless it first succeeds in reversing the inequality \( \kappa r > c \). Notice that \( \kappa \) appears in the linear analysis only indirectly, if at all, through \( F_1 = F'(Y^*) \).

   e) Large buffers are required to permit asymptotically stable constant solutions. Furthermore, the transient behavior of the system may be very demanding of resources.

4. **More Complex Systems: Preliminary results.**

   The system
\[
\frac{dX(t)}{dt} = a \left(1 - \frac{X(t)}{K}\right) - X(t) \left(F_1 Y(t) + F_0 (r X(t - T) - c)\right),
\]
\[
\frac{dY(t)}{dt} = r X(t - T) - c, \text{ if } Y(t) > 0 \text{ or the sum is } > 0, \ 0 \text{ otherwise},
\]

appears amenable to a similar analysis. It incorporates information about the rate of change of the buffer in the squelching decision. The characteristic equation is more complex, but $K$ is computable within the resources available. Similar results appear to follow, although more extensive numerical work is necessary. Parameter regions exist for which subcritical bifurcations can occur.

**Bibliography.**


