Exam Time: 3hrs. Do 10 problems. Each problem is worth 10 points. 100 pts total.

Restrictions: do 4 of 1-5; 4 of 6-10; 1 of 11-13.

1. Regular Expressions and NFAs.
   a) (5 pts) Give a regular expression for the set of all strings over $\Sigma = \{0, 1\}$ having a number of ones which is a multiple of 3.
   b) (5 pts) Construct an NFA that recognizing the language defined by this regular expression. Use the labeled digraph method. Make sure the steps can be followed by the reader.

2. Regular Expressions and NFAs. Show that the language $L = \{0^m1^n \mid 0 \leq m \leq n\}$ is not regular. Use two distinct methods:
   a) (5 pts) Define the relation $R_L$ and show that $\text{Index}(L) = \infty$;
   b) (5 pts) State and apply the Pumping Lemma for Regular Languages.

3. CF Languages. Let $L_1$ and $L_2$ be context-free languages. Prove that $L_1 \cup L_2$ is context-free.

4. CF Languages. For each of the following languages, determine whether it is context-free or not and prove your answer.
   a) (5 pts) $\{a^ib^jc^k \mid i + j = k\}$
   b) (5 pts) $\{a^ib^jc^k \mid i < j < k\}$

5. CF Languages. For the language $S \rightarrow \epsilon \mid SS \mid aSb$, construct a PDA that accepts the language.

6. Turing Machines.
   a) (5 pts) Define a one-tape Deterministic Turing Machine as given by your textbook.
   b) (5 pts) Show that the function $\pi_2(x_1, x_2, x_3)$ is Turing computable (by describing the Turing machine: make sure you have enough detail).

7. Turing Machines. (8 pts) Construct a multi-tape DTM to accept the language $\{w \mid w \in \{a, b\}^*, x = x^R\}$. (2 pts) What can you say about the time-complexity of your solution as compared to a one-tape Turing Machine using the same algorithm?
8. **Turing Machines.** Prove the Theorem: For any one-tape DTM $M = (Q, \Sigma, \Gamma, \delta, s)$, there exists a grammar $G_M = (V, \Sigma, R, S)$, with $V = Q \cup (\Gamma \setminus \Sigma) \cup \{[\cdot]\}$, such that for any configurations $(q, xay)$ and $(p, x'by')$, where $p, q \in Q, a, b \in \Gamma$, and $x, y, x', y' \in \Gamma^*$,

$$(q, xay) \vdash^* M (p, x'by') \text{ if and only if } [xqay] \Rightarrow^* G_M [x'pby']$$

9. **Primitive Recursive Functions.** Assume that $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is primitive recursive. Show that the function

$$g(n_1, \ldots, n_k, m) = \sum_{i=0}^{m} f(n_1, \ldots, n_k, i)$$

is also primitive recursive.

10. **R.E. and R. sets.** Prove that a nonempty set is recursive if and only if both it and its complement are recursively enumerable.

11. **Reducibility.**

   a) (3 pts) Define $m$-to-one reducible.

   b) (7 pts) Assume that $A \cup B = \{0, 1\}^*$ and that $A \cap B \neq \emptyset$. Show that, if $A$ and $B$ are r.e., then $A \leq_m A \cap B$.

12. **Context-Sensitive Languages.**

   a) (3 pts) Define $NSPACE(n)$.

   b) (7 pts) Using the given that Context-Sensitive Languages make up the complexity class $NSPACE(n)$, prove that the class of CSLs is closed under concatenation. Include enough detail so I know you know what you are doing.

13. **Complexity.**

   a) (2 pts) Define the class $NP$.

   b) (2 pts) Define *polynomial reducibility*.

   c) (2 pts) Define the problem $HC$.

   d) (4 pts) Prove that $HC \in NP$. 

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