NAME:____

ALGORITHMS QUALIFYING EXAM

This exam is open books and notes
and *closed* neighbors and calculators

The *upper bound* on exam time is 3 hours.

DO ALL OF PROBLEMS 1-6 AND EITHER PROBLEM 7 OR PROBLEMS 8-10.

Please put all your work on the exam paper.

(Partial credit will only be given if your work is shown.)

If you use pseudocode from a reference, such as the Cormen *et al*. textbook, you can just give the page reference instead of writing that pseudocode on your exam paper.

Your pseudocode should use the Cormen et al. conventions.

All algorithms referred to here use a sequential model of computation, not a parallel one.

Good luck!

DO ALL OF PROBLEMS 1-6

1: (18 points) NP-Completeness

Given:

- *m* real *x*-coordinates $x_1, x_2, \ldots, x_m$ and 3 *y*-coordinates y_1, y_2, y_3
- a set *S* of 3*m* points with coordinates $(x_i, y_j), \quad 1 \le i \le m, \quad 1 \le j \le 3$
- a collection *C* of (*not necessarily all*) triples of points from *S.*

Question: Does there exist $C' \subseteq C$ such that every point of *S* appears in exactly one triple of *C*'?

Prove that this problem is NP-complete.

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2: (18 points) Train Conductor

A cargo train conductor is responsible for planning train station stops along railroad tracks from a starting station to the station at the end of the route. The train has sufficient resources to travel *n* miles without stopping. The conductor knows the distances between train stations along the route. The conductor wants to make as few stops as possible. Give an efficient algorithm by which he/she can determine at which train stations to stop.

Make sure that you provide the following for your algorithm:

(a) pseudocode

(b) worst-case asymptotic running time analysis

(c) proof that your algorithm yields an optimal solution.

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3: (18 points) Amortized Analysis for Heap

Consider a binary heap of size n in which the minimum-valued element is stored at the root. Recall that the worst-case asymptotic cost of a heap insertion operation is in O(lgn). The worst-case asymptotic cost of a heap deletion operation is also in O(lgn).

Provide a valid potential function so that the amortized cost of heap insertion is in O(lgn) but the amortized cost of deleting the smallest element is in O(1). *Justify your answer*.

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4: (18 points) Network Flows

Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted αf, is a function from $V \times V$ to R defined by
$(\alpha f)(u, v) = \alpha \cdot f(u, v)$.

Prove that the flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \le \alpha \le 1$.

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5: (5 points) Asymptotic Growth of Functions

(a) (3 points) Given the 3 following functions:

$\lg^4 n$ $\lg^{-5} n$ $4^{\lg \lg n}$

List them below in nondecreasing asymptotic order of growth:

1) *smallest* **2)** **3)** *largest*

Now, given the following 3 facts about some unknown functions $f_1(n), f_2(n), f_3(n)$:

1) $f_1(n) \in \Omega(\lg^4 n)$ 2) $f_2(n) \in O(4^{\lg \lg n})$ 3) $f_3(n) \in \Theta(\lg^{-5} n)$

Circle TRUE or FALSE for each statement below & briefly justify your choice.

(b) (1 point) $f_2(n) \in \Omega(f_3(n))$

TRUE **FALSE**

(c) (1 point) $f_2(n) \in O(f_1(n))$

TRUE **FALSE**

6: (5 points) Recurrence

Find a tight upper and lower bound on the closed-form solution for the following recurrence:

$$T(n) = \begin{cases} 6T\left(\frac{n}{3}\right) + 2^{3\lg n} & n > k \\ \Theta(1) & n \le k \end{cases}$$

where *k* is a small integer. That is, find a function $g(n)$ such that $T(n) \in \Theta(g(n))$.

DO EITHER PROBLEM 7 OR ALL OF PROBLEMS 8-10

7: (18 points) Linear Equations

Consider the following system of 4 linear inequalities in 4 variables:

$$x_1 - x_2 \leq 1$$
$$x_2 - x_3 \leq -2$$
$$x_3 - x_4 \leq 3$$
$$x_4 - x_1 \leq -4$$

(a) Find a feasible solution or determine that no feasible solution exists.

(b) What algorithm did you use to solve the problem?

(c) Analyze the worst-case asymptotic running time of the algorithm that you used. Express the running time in general terms as a function of the number of equations *m* and the number of variables *n*.

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8: (5 points) Approximation Algorithms

(circle TRUE or FALSE for the statement below and justify your choice)

TRUE **FALSE**

For a maximization problem, if we know that:

$$C \le \frac{C'}{\rho(n)} \quad \text{and} \quad C \le C^* \le C'$$

where *n* is the size of the input, *C* is the cost value returned from an algorithm and *C** is the optimal cost for the maximization problem, then we can conclude that the algorithm is a $\rho(n)$-approximation algorithm for the problem.

9: (5 points) Number-Theoretic Algorithms

In the RSA cryptographic system, suppose Alice encrypts a message M using Bob's public key P=(e,n) to form P(M). Next, Alice performs C = $(P(M))^2$ (mod n). Then, Alice sends C to Bob. Can Bob successfully use his secret key S=(d,n) to decode Alice's message to obtain M? Explain your answer.

10: (5 points) Strings

Let: $|x|$ denote the length of character string *x*

$w > x$ signify that *w* is a prefix of *x*

$w < x$ signify that *w* is a suffix of *x*

(circle TRUE or FALSE for the statement below and justify your choice)

TRUE FALSE

For 3 character strings *a*, *b*, and *c*:

$c < b$ and $b < a$ and $c > a$ imply that $|b| = 0$.