Foundations Qualifying Examination, Spring 2004

This exam is closed book. Complete as many problems as you can. Justify all your answers. You may do the problems in any order, but start each problem on a new page and label the problem. Show your work, as partial credit may be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

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1. (10 points) For a string x and a character y, let $n_y(x)$ be the number of y characters in x. Let $\Sigma = \{a, b, c\}$ and $L = \{\, x \in \Sigma^* \mid n_a(x) \equiv n_b(x) \equiv n_c(x) \pmod 2 \,\}$. In other words, each string in L has the same parity of a, b, and c characters. So $bbacb \in L$ and $aa \in L$ but $cbb \notin L$.

   Let $\sim$ be the equivalence relation over Σ^* such that $x \sim y$ exactly when for all z, $xz \in L \iff yz \in L$. How many equivalence classes does $\sim$ have? Explain the significance of this result.

2. (10 points) Prove that no regular language is inherently ambiguous.

3. (10 points) Suppose $M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_1, F_1)$ is a nondeterministic PDA and that no computation of M_1 on any input string causes its stack to hold more than 3 elements. It follows that $L(M_1)$ is regular. Prove this by formally constructing an an NFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ in terms of M_1 such that $L(M_1) = L(M_2)$. You may use any PDA semantics that are reasonable and convenient, but you must state what they are either by listing a textbook author name as a source (preferred), or commenting your construction so that a reader can tell what the semantics are.

4. Let $\Sigma = \{a, b, c\}$ and $L = \left\{ a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k \right\}$.

   (a) (6 points) Prove that L is not regular. (Due the the next property you are asked to prove, it is unwise to attempt a pumping lemma argument in this case.)

   (b) (6 points) Let $\phi(p)$ be the property that for every string $s \in \Sigma^*$, if $s \in L$ and $|s| \geq p$ then there exists strings $x, y, z \in \Sigma^*$ such that $s = xyz$ and $|y| > 0$ and $|xy| \leq p$ and for all $i \geq 0$, $xy^iz \in L$. (In other words, $\phi(p)$ means that strings of length p and greater are p-regular-pumpable in L.) Prove that $\phi(2)$ holds.

   (c) (8 points) Demonstrate a string that is a counterexample to $\phi(1)$.

5. (15 points) Given two Turing machines M_1 and M_2, is it decidable whether $L(M_1) \subseteq L(M_2)$? Defend your conclusion.

6. (a) (10 points) Given a directed graph $G = (V, E)$ and a pair of nodes $u, v \in V$. Show that determining whether there is a directed path in G from u to v can be carried out by a deterministic Turing machine in $O(\log^2 |V|)$ space.

   (b) (10 points) Use result 6(a) to show that an n-space bounded nondeterministic Turing machine can be simulated by a $O(n^2)$-space bounded deterministic Turing machine.

7. (15 points) Instances of kSAT are k-CNF formulas, in which each clause consists of exactly k different literals and a literal ℓ and its complement $\neg\ell$ cannot occur in one clause. 3SAT has been proven NP-complete. Use this fact to show that 4SAT is also NP-complete.