Foundations Qualifying Examination, Spring 2003

This exam is closed book. All problems are equally weighted. Complete as many problems as you can. Justify all your answers. You may do the problems in any order, but start each problem on a new page and label the problem. Show your work, as partial credit may be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

Problem	Points	Grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	**100**	

1. Let L be a regular language. Show that there exists a positive integer n such that for every string $w \in L$, if $|w| \geq n$, then there exist strings x, y, and z such that $w = xyz$, $|xy| \leq n$, $|y| > 0$, and $xz \in L$.

2. Select only one:

   (a) Let L be a language over Σ. Let $L_{\frac{1}{2}} = \{x \mid (\exists y)[|x| = |y| \text{ and } xy \in L]\}$. Show that if L is regular, then so is $L_{\frac{1}{2}}$.

   (b) Let $M_i = (Q_i, \Sigma, \delta_i, s_i, F_i)$, $i \in \{0, 1\}$, be two NFA's. Construct a new NFA $M = (Q, \Sigma, \delta, s, F)$ such that $L(M) = L(M_1) \cap L(M_2)$. You must provide detailed descriptions of Q, δ, s, and F.

   (c) Let L be a regular language over Σ, and

   $$R_L = \{(x, y) \mid x, y \in \Sigma^* \text{ and } (\forall w \in \Sigma^*)[xw \in L \text{ if and only if } yw \in L]\}.$$

   Show that R_L is an equivalence relation and that the number of equivalence classes in R_L is finite.

3. Let L be a language over $\{a, b\}$, where each element contains an even number of a's and even number of b's. Construct a context-free grammar G that generates L. Is L regular? Justify your answer.

4-5. Let x be a string over $\{a, b\}$. Denote by $a(x)$ the number of a's contained in x, and $b(x)$ the number of b's contained in x. Consider the following two languages:

$$\begin{aligned} L_1 &= \{x \in \{a, b\}^* \mid a(x) < b(x)\} \\ L_2 &= \{x \in \{a, b\}^* \mid a(x) = b^2(x)\} \end{aligned}$$

If a language is context-free, describe in English a PDA that accepts it. If a language is not context-free, provide proof why it is not context-free.

6. Show that if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$; where $\leq_m$ denotes many-one reductions.

7. Let $L = \{\langle M \rangle : M \text{ is a TM and } 0 \in L(M)\}$. Is L decidable? Justify your answer.

8. Show how to simulate a one-tape nondeterministic Turing machine using a 3-tape deterministic Turing machine.

9. Show that if L can be accepted by a Turing machine, then L can be recursively enumerated by a deterministic Turing machine, where each element of L is enumerated exactly once.

10. Describe and show that there is a tiling problem that is undecidable.