Foundations Qualifying Examination, Fall 2004

This exam is closed book. Complete as many problems as you can and justify your answers. You may do the problems in any order, but start each problem on a new page and label the problem. It is important that you show your work for the possibility of getting partial credit. You will be graded not only on the correctness of your answer, but also on the clarity you express it.

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1. (5 points) Define the quotient of two languages L_1 and L_2 as

$$L_1/L_2 = \{\, x \mid (\exists y \in L_2)\,[xy \in L_1] \,\}\,.$$

   Let $A = 0\{0,1\}^*0 \cup 1\{0,1\}^*1$ and $B = \{\, 0^n1^n \mid n \geq 0 \,\}$. Draw a DFA that accepts A/B.

2. Let $A = \{\, x \in \{0,1\}^* : |x| \geq 1 \text{ and the binary number } x \text{ is divisible by } 3 \,\}$. For example, $0110 \in A$ but $100 \notin A$.

   (a) (5 points) Let $\sim$ be the equivalence relation over $\{0,1\}^*$ such that $x \sim y$ exactly when for all z, $xz \in A \iff yz \in A$. What are the equivalence classes of $\sim$ in this case?

   (b) (5 points) Draw a minimum-state DFA that accepts A.

3. (10 points) Prove that $L = \{\, 0^{pq} \mid p \text{ and } q \text{ are prime} \,\}$ is not regular.

4. (5 points) Prove that the class of context-free languages is not closed under intersection.

5. Let G be the context-free grammar with rules

$$S \rightarrow AaSbB \mid \varepsilon, \quad A \rightarrow aA \mid a, \quad B \rightarrow bB \mid \varepsilon$$

   (a) (5 points) Show that G is ambiguous.

   (b) (5 points) Write an equivalent unambiguous grammar for G.

6. (10 points) Construct a nondeterministic PDA that accepts the language $L = \{\, a^ib^j \mid 2i \neq j \,\}$. Use the convention that the PDA does *not* need to have an empty stack in order to accept: it only needs to be able to reach an accepting state.

7. (17 points) Assume that $A \cup B = \{0,1\}^*$, $A \cap B \neq \emptyset$, and A and B are recursively enumerable. Show that $A \leq_m A \cap B$. That is, show that there is a recursive function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that $(\forall x)[x \in A \text{ iff } f(x) \in A \cap B]$.

8. (17 points) Show that if $A \in DTIME(n)$, then $A^* \in DTIME(n^3)$, where A^* is the Kleene closure of A.

9. (16 points) For any given NTM M with a binary input alphabet, define a bounded halting problem $BH(M)$ as follows: Given an instance (x, y), where x and y are binary strings, decide whether M accepts x in $|y|$ steps. (Note: $|y|$ denotes the length of y.) Show that there exists an NTM M' such that $BH(M')$ is NP-complete.