## ***Counting Methods and the Pigeonhole Principle***

Basic Principles :

**Multiplication Principle** :

If an activity can be constructed in t successive steps and step1 can be done in $n_1$ ways ; step 2 can be done in $n_2$;... and step t can be done in $n_t$ ways

then the number of different possible activities is

$$n_1 * n_2 ... * n_t \quad .$$

## Example

a. How many strings of length 4 can be formed using the letters ABCDE if repetitions are not allowed ?

Think of this as a problem of filling 4 slots ( in turn)

___ ___ ___ ___ with choices from ABCDE and no repeats are allowed.

_5_ _4_ _3_ _2_ = 120 choices

b. How many strings begin with the letter B ?

_1_ _4_ _3_ _2_ = 24

c. How many do not begin with the letter B ?

_4_ _4_ _3_ _2_ = 96 ( = 120 – 24 )

## Addition Principle :

Suppose $X_1,X_2,...,X_t$ are sets and that the ith set $X_i$ has $n_i$ elements. If $\{ X_1,X_2,...,X_t \}$ is a pairwise disjoint family, the number of possible elements that can be selected from $X_1$ or $X_2$ ... or $X_t$ is

$$n_1 + n_2 + ... + n_t .$$

( or, restated, the union of all $\{X_1,X_2,...,X_t\}$ contains

$n_1 + n_2 + ... + n_t$ elements )

## ***Permutations and Combinations***

Defn: A ***permutation*** of n distinct elements $x_1, x_2,...,x_n$ is an ordering of the n elements $x_1, x_2,...,x_n$

***Theorem:*** There are n! permutations of n elements.

Proof: Use multiplication principle and fill in the n blanks to get n.(n-1)...3.2.1 = n !

**Defn.** : An ***r-permutation*** of n (distinct) elements $x_1, x_2, \ldots, x_n$ is an ordering of an r-element subset of $\{ x_1, x_2, \ldots, x_n \}$ .

The number of r-permutations of a set of n distinct elements is denoted, **P(n,r)** .

**Theorem**: The number of r-permutations of a set of n distinct objects is

P ( n , r ) = n (n-1) ( n-2) ... ( n - r +1) for $r \leq n$ .

Pf : Multiplication principle.

P ( n , r ) =

$$= n \bullet (n-1) \bullet (n-2) \bullet \ldots (n-r+1)$$

$$= \frac{n \bullet (n-1) \bullet (n-2) \bullet \ldots (n-r+1) \bullet (n-r) \bullet \ldots 3 \bullet 2 \bullet 1}{(n-r) \bullet (n-r-1) \bullet \ldots 3. \bullet 2. \bullet 1}$$

$$= \frac{n!}{(n-r)!}$$

So , P ( n , r ) = n ! / ( n -r ) !

Defn. : Given a set X = { x1,x2,..., xn } of n distinct elements

a) an ***r -combination of X*** is an *unordered selection of r elements of X* ( an r-element subset of X )

b) The number of r-combinations of a set of n distinct elements is denoted by **C ( n , r )** or $\binom{n}{r}$.

P ( n , r ) = r ! C ( n , r )

***Theorem*** : The number of r-combinations of n distinct objects is

C ( n , r ) = P ( n , r ) / r !

$$= \frac{n!}{(n-r)!\, r!} \quad , r \leq n \quad .$$

## *Generalized Permutation and Combinations*

Now consider ordering of sequences with repetitions and unordered selections with repetitions.

**Example** : consider the 11 letter word : Mississippi

How many strings can be formed with the letters?

With 11 spaces to fill ( this implies order is being considered) by selecting from : 4 -s ; 4 - i, 2 - p , 1- m.

The problem is a combinatoric one :

so, using multiplication principle,

= ( # ways can choose to locate the p's in the 11 places) x (# ways can choose to locate the s's in the 9 spaces remaining) x (# ways can choose to locate the I's in the remaining 5 places) x (# ways can choose to locate the m in the remaining 1 space)

= C( 11,2) C(9,4) C(5,4) C(1,1) $= \frac{11!}{2!9!} \frac{9!}{5!4!} \frac{5!}{4!1!} 1 =$ 34,650.

## Binomial Coefficients and Combinatorial Identities

Think of $(a+b)^n$ as the n - product of ( a+b)

and the expansion can be obtained by making n choices of an a or b from each factor, multiplying them, in turn, and adding all the results then what we get for coefficients can be seen as

$$(a+b)^n = C(n,0)\, a^n b^0 + C(n,1) a^{n-1} b^1 + C(n,2) a^{n-2} b^2 + \ldots + C(n,n-1) a^1 b^{n-1} + C(n,n)\, a^0 b^n$$

***Theorem* : *Binomial Theorem***

If a and b are real numbers, n a positive integer then

$$(a+b)^n = \sum_{k=0}^{n} C(n,k) a^{n-k} b^k$$

The terms, C(n,k) are called **binomial coefficients.**

When the binomial coefficients are arranged in a descending and expanding triangle with increasing n, the generated triangle is known as **Pascal's triangle**.

The borders are all 1's the internal entries are defined by the combinatorial identity

**Theorem** : C ( n + 1 , k ) = C ( n , k - 1 ) + C ( n , k ) for $1 \leq k \leq n$ .

Proof : Let X be a set with n elements. Choose and element a not in X. Then has n+1 elements and C ( n+1, k ) counts the number of k-element subsets of Y - some of which contain a , some of which do not contain a . Those that do not contain a are generated from a set of size n and so there are C ( n , k ) of them.

The others have a as an element and k-1 other elements chosen from n elements , there are C ( n, k-1 ) such sets.

Hence :

C ( n + 1 , k ) = C ( n , k - 1 ) + C ( n , k )

## The Pigeonhole Principle

***Pigeonhole Principle* ( First Form)**

**If n pigeons fly into k pigeonholes and k < n , then some pigeonhole has at least two pigeons.**

***Pigeonhole Principle* ( Second Form)**

If a function $f : X \longrightarrow Y$ , with $\infty > |X| > |Y|$ , then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$ and $x_1 \neq x_2$ .

Proof : since X and Y are both finite sets, let X be # pigeons and Y = # pigeonholes and apply the First Form.

***Pigeonhole Principle* (Third Form)**

Let $f$ be a function : $X \longrightarrow Y$ , with $\infty > |X| > |Y|$

Suppose that | X | = n and | Y | = m. Let $k = \left\lceil \frac{n}{m} \right\rceil$ .

Then there are at least k values $a_1, a_2, \ldots, a_k \in X$ such that

$f(a_1) = f(a_2) = \ldots = f(a_k)$.