DFA additional formal definitions

- multistep transition function $\hat{\delta}$
  a.k.a. extended transition function

$$\hat{\delta}(q, \epsilon) = q$$
$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

- Language recognized by a DFA:

$$\mathcal{L}(D) = \{ w \mid \hat{\delta}(q_0, w) \in F \}$$

initial state of D

- Regular languages

$$L \text{ is regular} \equiv \exists \text{ DFA } D \text{ s.t. } \mathcal{L}(D) = L$$

NFA additional definitions

- multistep transition function

$$\hat{\delta}(q, \epsilon) = \epsilon\text{-close}(\{q\})$$

$$\hat{\delta}(q, xa) = \epsilon\text{-close}\Big(\bigcup_{r \in \hat{\delta}(q,x)} \delta(r, a) \Big)$$

- ϵ-close

$\epsilon\text{-close}(R) =$

the least set s.t. - $R \subseteq \epsilon\text{-close}(R)$

- and $\forall q \in R \;\; \delta(q, \epsilon) \subseteq \epsilon\text{-close}(R)$

- Language recognized by an NFA

$$\mathcal{L}(N) = \{ w \mid (\hat{\delta}(q_0, w) \cap F) \neq \emptyset \}$$

thm. $\boxed{\text{if } L_1, L_2 \text{ are regular then } L_1 \cup L_2 \text{ is regular.}}$

Proof

L_1 regular $\Rightarrow \exists D_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ s.t. $\mathcal{L}(D_1) = L_1$

L_2 regular $\Rightarrow \exists D_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ s.t. $\mathcal{L}(D_2) = L_2$

Without loss of generality, we assume same alphabet.
(two DFAs could be transformed to work on same Σ)

Question: $\exists D = (Q, \Sigma, \delta, q_0, F)$ s.t. $\mathcal{L}(D) = L_1 \cup L_2$?

we use the DFA of the product construction seen in class.

$\mathcal{L}(D) = \mathcal{L}(D_1) \cup \mathcal{L}(D_2) =$

$\{w \mid \hat{\delta}(q_0, w) \in F\} = \{w \mid \hat{\delta}_1(q_1, w) \in F_1\} \cup \{w \mid \hat{\delta}(q_2, w) \in F_2\}$

$q_0 = (q_1, q_2)$ in the construction.

$F = \{(q, q') \mid q \in F_1 \vee q' \in F_2\}$ in the construction

$\{w \mid \hat{\delta}((q_1, q_2), w) \in F\} = \{w \mid (\hat{\delta}_1(q_1, w), \hat{\delta}_2(q_2, w)) \in F\}$

Therefore we simply need to prove

Lemma. $\hat{\delta}((q_1, q_2), w) = (\hat{\delta}_1(q_1, w), \hat{\delta}(q_2, w))$

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Lemma.

$$\hat{\delta}((q_1,q_2), w) = (\hat{\delta}_1(q_1, w), \hat{\delta}_2(q_2, w))$$

Proof by induction on the structure of w.

- base case: ϵ

$$\hat{\delta}((q_1,q_2), \epsilon) = (q_1, q_2)$$

$$(\hat{\delta}_1(q_1,\epsilon), \hat{\delta}_2(q_2,\epsilon)) = (q_1,q_2)$$

✓

- inductive case: $\overset{\text{subterm}}{x}a$

IH = Inductive Hypothesys = $\hat{\delta}((q_1,q_2), x) = (\hat{\delta}_1(q_1,x)\ \hat{\delta}_2(q_2,x))$

Can I prove

$$\hat{\delta}((q_1,q_2), xa) = (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))\ ?$$

$=$

by Def of $\hat{\delta}$ $\quad \delta(\hat{\delta}((q_1,q_2),x), a)$

$=$

by IH $\quad \delta((\hat{\delta}_1(q_1,x), \hat{\delta}_2(q_2,x)), a)$

$=$

by construction of δ $\quad (\underbrace{\delta_1(\hat{\delta}_1(q_1,x), a)}, \underbrace{\delta_2(\hat{\delta}_2(q_2,x), a)})$

this is $\hat{\delta}_1(q_1, xa)$ by expanding the def. of $\hat{\delta}_1$

this is $\hat{\delta}_2(q_2, xa)$ by expanding the def of $\hat{\delta}_2$

Hence $\quad = (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa)) \leftarrow =$