Probabilistic Robotics

Lecture 2

Probabilities
Bayes rule
Bayes filters

Based on slides from book’s website
Probabilistic Robotics

Key idea: Explicit representation of uncertainty using the calculus of probability theory

- Perception = state estimation
- Action = utility optimization
Axioms of Probability Theory

Pr(A) denotes probability that proposition A is true.

- \(0 \leq \Pr(A) \leq 1\)
- \(\Pr(True) = 1\) \(\Pr(False) = 0\)
- \(\Pr(A \lor B) = \Pr(A) + \Pr(B) - \Pr(A \land B)\)
A Closer Look at Axiom 3

\[ \text{Pr}(A \lor B) = \text{Pr}(A) + \text{Pr}(B) - \text{Pr}(A \land B) \]
Using the Axioms

\[
\begin{align*}
\Pr(A \lor \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \land \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{align*}
\]
Discrete Random Variables

• $X$ denotes a random variable.
• $X$ can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
• $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable $X$ takes on value $x_i$.
• $P(\cdot)$ is called probability mass function.

• E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
Continuous Random Variables

- $X$ takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in (a,b)) = \int_a^b p(x) \, dx$$

- E.g.
Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$

- If $X$ and $Y$ are independent then
  
  $P(x,y) = P(x) \cdot P(y)$

- $P(x \mid y)$ is the probability of $x$ given $y$
  
  $P(x \mid y) = P(x,y) / P(y)$
  
  $P(x,y) = P(x \mid y) \cdot P(y)$

- If $X$ and $Y$ are independent then
  
  $P(x \mid y) = P(x)$
Law of Total Probability, Marginals

Discrete case

\[ \sum_x P(x) = 1 \]

\[ P(x) = \sum_y P(x, y) \]

\[ P(x) = \sum_y P(x \mid y)P(y) \]

Continuous case

\[ \int p(x) \, dx = 1 \]

\[ p(x) = \int p(x, y) \, dy \]

\[ p(x) = \int p(x \mid y) p(y) \, dy \]
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ \Rightarrow \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]

posterior
Normalization

\[ P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \eta P(y \mid x) P(x) \]
\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)} \]

Algorithm:

\[ \forall x: \text{aux}_{x \mid y} = P(y \mid x) P(x) \]
\[ \eta = \frac{1}{\sum_x \text{aux}_{x \mid y}} \]
\[ \forall x: P(x \mid y) = \eta \text{aux}_{x \mid y} \]
Conditioning

• Law of total probability:

\[
P(x) = \int P(x, z) dz
\]

\[
P(x) = \int P(x \mid z) P(z) dz
\]

\[
P(x \mid y) = \int P(x \mid y, z) P(z \mid y) dz
\]
Bayes Rule with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z) \cdot P(x \mid z)}{P(y \mid z)} \]
Conditional Independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

equivalent to

\[ P(x \mid z) = P(x \mid z, y) \]

and

\[ P(y \mid z) = P(y \mid z, x) \]
Simple Example of State Estimation

• Suppose a robot obtains measurement $z$
• What is $P(\text{open}|z)$?
Causal vs. Diagnostic Reasoning

- $P(\text{open} | z)$ is diagnostic.
- $P(z | \text{open})$ is causal.
- Often causal knowledge is easier to obtain. **count frequencies!**
- Bayes rule allows us to use causal knowledge:

$$P(\text{open} | z) = \frac{P(z | \text{open})P(\text{open})}{P(z)}$$
Example

- \( P(z|\text{open}) = 0.6 \quad P(z|\neg\text{open}) = 0.3 \)
- \( P(\text{open}) = P(\neg\text{open}) = 0.5 \)

\[
P(\text{open} | z) = \frac{P(z|\text{open})P(\text{open})}{P(z)}
\]

\[
P(\text{open} | z) = \frac{P(z|\text{open})P(\text{open})}{P(z|\text{open})P(\text{open}) + P(z|\neg\text{open})P(\neg\text{open})}
\]

\[
P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67
\]

- \( z \) raises the probability that the door is open.
Combining Evidence

• Suppose our robot obtains another observation $z_2$.

• How can we integrate this new information?

• More generally, how can we estimate $P(x \mid z_1...z_n)$?
Recursive Bayesian Updating

\[
P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
\]

**Markov assumption:** \(z_n\) is independent of \(z_1, \ldots, z_{n-1}\) if we know \(x\).

\[
P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
= \eta \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
= \eta \eta_1 \ldots n \prod_{i=1}^{n-1} P(z_i \mid x) \ P(x)
\]
Example: Second Measurement

- $P(z_2|\text{open}) = 0.5$ \hspace{1cm} $P(z_2|\neg\text{open}) = 0.6$

- $P(\text{open}|z_1)=2/3$

\[
P(\text{open} | z_2, z_1) = \frac{P(z_2 | \text{open}) P(\text{open} | z_1)}{P(z_2 | \text{open}) P(\text{open} | z_1) + P(z_2 | \neg\text{open}) P(\neg\text{open} | z_1)}
\]

\[
= \frac{1 \cdot 2}{2} \cdot \frac{3}{3} + \frac{5}{2} \cdot \frac{1}{3} = \frac{5}{8} = 0.625
\]

- $z_2$ lowers the probability that the door is open.
Actions

• Often the world is *dynamic* since
  • *actions carried out by the robot*,
  • *actions carried out by other agents*,
  • or just the *time* passing by

• How can we *incorporate* such *actions*?
Typical Actions

• The robot turns its wheels to move
• The robot uses its manipulator to grasp an object
• Plants grow over time...

• Actions are never carried out with absolute certainty.
• In contrast to measurements, actions generally increase the uncertainty.
Modeling Actions

Control system

World model, belief

Perceptual/action data

Environment, state

Actions
Modeling Actions

World model, belief

Control system

Perceptual/action data

Environment, state

Actions

U
Modeling Actions and the World
Modeling Actions

• To incorporate the outcome of an action $u$ into the current “belief”, we use the conditional pdf

$$P(x|u,x')$$

• This term specifies the pdf that executing $u$ changes the state from $x'$ to $x$. 
Example: Closing the door
State Transitions

\[ P(x|u,x') \text{ for } u = "close door": \]

If the door is open, the action “close door” succeeds in 90% of all cases.
Integrating the Outcome of Actions

Continuous case:

\[ P(x \mid u) = \int P(x \mid u, x') P(x') dx' \]

Discrete case:

\[ P(x \mid u) = \sum P(x \mid u, x') P(x') \]
Example: The Resulting Belief

\[ P(\text{closed} \mid u) = \sum P(\text{closed} \mid u, x') P(x') \]

\[ = P(\text{closed} \mid u, \text{open}) P(\text{open}) \]

\[ + P(\text{closed} \mid u, \text{closed}) P(\text{closed}) \]

\[ = \frac{9 \times 5 + 1 \times 3}{10 \times 8 + 1 \times 8} = \frac{15}{16} \]

\[ P(\text{open} \mid u) = \sum P(\text{open} \mid u, x') P(x') \]

\[ = P(\text{open} \mid u, \text{open}) P(\text{open}) \]

\[ + P(\text{open} \mid u, \text{closed}) P(\text{closed}) \]

\[ = \frac{1 \times 5 + 0 \times 3}{10 \times 8 + 1 \times 8} = \frac{1}{16} \]

\[ = 1 - P(\text{closed} \mid u) \]
Bayes Filters: Framework

• **Given:**
  • Stream of observations $z$ and action data $u$:
    \[ d_t = \{u_1, z_1, \ldots, u_t, z_t\} \]
  • Sensor model $P(z|x)$.
  • Action model $P(x|u, x')$.
  • Prior probability of the system state $P(x)$.

• **Wanted:**
  • Estimate of the state $X$ of a dynamical system.
  • The posterior of the state is also called **Belief**:
    \[ Bel(x_t) = P(x_t | u_1, z_1, \ldots, u_t, z_t) \]
Markov Assumption

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

\[ p(z_t | x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t | x_t) \]
\[ p(x_t | x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t | x_{t-1}, u_t) \]
Bayes Filters

\[
Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)
\]

Bayes

\[
= \eta P(z_t \mid x_t, u_1, z_1, \ldots, u_t) P(x_t \mid u_1, z_1, \ldots, u_t)
\]

Markov

\[
= \eta P(z_t \mid x_t) P(x_t \mid u_1, z_1, \ldots, u_t)
\]

Total prob.

\[
= \eta P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1})
P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}
\]

Markov

\[
= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}
\]

Markov

\[
= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \, dx_{t-1}
\]

\[
= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) \, dx_{t-1}
\]

z = observation
u = action
x = state
Algorithm **Bayes_filter** ($Bel(x), d$):

1. $\eta = 0$
2. If $d$ is a perceptual data item $z$ then
3. For all $x$ do
4. $Bel'(x) = P(z \mid x)Bel(x)$
5. $\eta = \eta + Bel'(x)$
6. For all $x$ do
7. $Bel'(x) = \eta^{-1}Bel'(x)$
8. Else if $d$ is an action data item $u$ then
9. For all $x$ do
10. $Bel'(x) = \int P(x \mid u, x') Bel(x') \, dx'$
11. Return $Bel'(x)$

$$Bel(x_t) = \eta \cdot P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \cdot Bel(x_{t-1}) \, dx_{t-1}$$
Bayes Filters are used in many places

\[ Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) \, dx_{t-1} \]

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)
Summary

• Bayes rule allows us to compute probabilities that are hard to assess otherwise.

• Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.

• Bayes filters are a probabilistic tool for estimating the state of dynamic systems.